Limits, continuity and differentiability

Limit:

Suppose $f(x)$ is defined when x is near the number "a" then we write

$$\lim_{x \to a} f(x) = L.$$

Then the fn. $f(x)$ is said to have limit value "L" as x tends to "a".

Left limit: when $x < a$, $x \to a$, $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$

Right limit: when $x > a$, $x \to a$, $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$

Important results in Limit:

(i) $\lim_{x \to a} f(x) = L$ iff $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$.

A limit exists if L.H.L = R.H.L.

Example

(i) If $f(x) = |x|$ then $\lim_{x \to 0} |x| = ?$

Soln

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Since $|x| = x$, for $x > 0$, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 = L.H.L$$

for $x < 0$, $|x| = -x$, we have

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = 0 = R.H.L$$

$$R.H.L = L.H.L$$

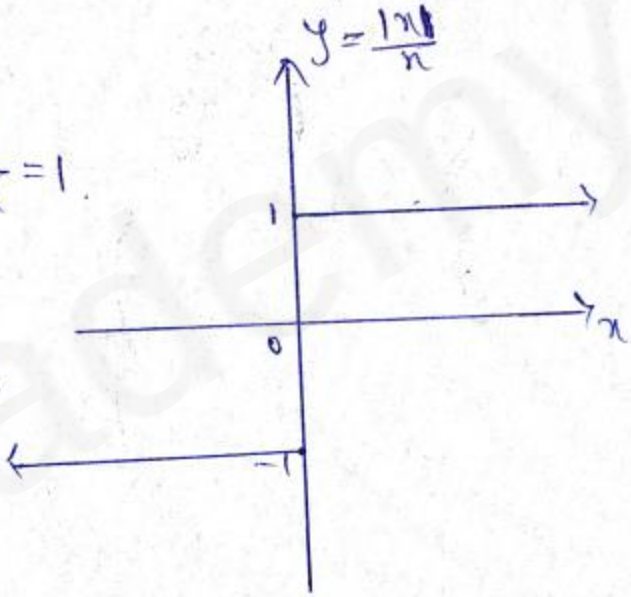
$$\therefore \lim_{x \to 0} |x| = 0$$

(ii) If $f(x) = \frac{|x|}{x}$ then $\lim_{x \to 0} \frac{|x|}{x} = ?$

Soln

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0} \frac{x}{x} = 1.$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0} \frac{-x}{x} = -1.$$



L.H.L $\neq$ R.H.L.

$\therefore \lim_{x\to 0} \frac{|x|}{x}$ does not exists.

Result (ii)

If f is a polynomial (or) a ~~fn~~ rational fn. and 'a' is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Result (iii)

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

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Indeterminate form: $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 1^{\infty}, 0^{0}, \infty^{0}$

L' Hospital Rule:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Limit laws:

(1) $\lim_{x\to a}\left(f(x)\pm g(x)\right) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$.

(2) $\lim_{x\to a} c\, f(x) = c \lim_{x\to a} f(x)$, c is a constant.

(3) $\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$. ③

(4) $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$.

(5) $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n$, n is +ve integer.

(6) $\lim_{x \to a} c = c$

(7) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, n is a +ve integer

Standard limits:

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

(ii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(iv) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

(v) $\lim_{x \to 0} \frac{e^{mx} - 1}{x} = e$

(vi) $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

(vii) $\lim_{x \to 0} \frac{\sin mx}{x} = m$

(viii) $\lim_{x \to 0} [1 + ax]^{1/x} = e^a$

(ix) $\lim_{x \to 0} \left[1 + \frac{a}{x}\right]^x = e^a$

(x) $\lim_{x \to 0} \left[\frac{a^x + b^x}{2}\right]^{1/x} = \sqrt{ab}$

(xi) $\lim_{x \to 0} \left[\frac{1 - \cos ax}{x^2}\right] = \frac{a^2}{2}$

(xii) $\lim_{x \to \infty} \frac{\log_e x}{x} = 0$

(xiii) $\lim_{x \to 0} [\cos x + a \sin bx]^{1/x} = e^{ab}$

(4)

2020-CE

Problems

(1) The value of $\lim_{x\to\infty} \frac{x^2-5x+4}{4x^2+2x}$ is ____

Soln

$$\lim_{x\to\infty} \frac{x^2-5x+4}{4x^2+2x} = \lim_{x\to\infty} \frac{x^2\left(1-5/x+4/x^2\right)}{x^2\left(4+2/x\right)}$$

$$= \lim_{x\to\infty} \frac{1-5/x+4/x^2}{4+2/x}.$$

$$= \frac{1-0+0}{4+0} = 1/4$$

(2) Let $L = \lim_{x \to \pi/2} (\sin x)^{\tan x}$. The value of L is ___ $\boxed{2020\text{-TF}}$

Soln

$$\lim_{x \to \pi/2} (\sin x)^{\tan x} \to 1^{\infty} \text{ (indeterminate form)}$$

Take $y = (\sin x)^{\tan x}$

$$\log y = \log (\sin x)^{\tan x}$$

$$\log y = \tan x \log (\sin x)$$

$$\lim_{x \to \pi/2} \log y = \lim_{x \to \pi/2} \frac{\log(\sin x)}{\cot x} \quad \left(\tfrac{0}{0}\right)$$

$$= \lim_{x \to \pi/2} \frac{\frac{1}{\sin x} (\cos x)}{-\operatorname{cosec}^2 x} \quad (\because \text{L'Hos rule})$$

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$$= 0.$$

$$\lim_{x \to \pi/2} y = e^0 = 1$$

$$\lim_{x \to \pi/2} (\sin x)^{\tan x} = 1$$

(3) The value of the expression [2019 - CH]

$$\lim_{x \to \pi/2} \left| \frac{\tan x}{x} \right| \text{ is}$$

Soln

$$\lim_{x \to \pi/2} \left| \frac{\tan x}{x} \right| = \infty \quad (\because \tan \pi/2 = \infty)$$

(or)

limit does not exists.

(4) The value of the following limit is ___ [2019 - AE]

$$\lim_{\theta \to 0} \frac{\theta - \sin\theta}{\theta^3}.$$

Soln

$$\lim_{\theta \to 0} \frac{\theta - \sin\theta}{\theta^3} \quad (0/0 \text{ form}).$$

$$= \lim_{\theta \to 0} \frac{1 - \cos\theta}{3\theta^2} \quad (\because \text{L' Hos rule}) \rightarrow (0/0)$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{6\theta} = 1/6 \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1/6 (1) = 1/6$$

6.

5) If $y = e^{-x^2}$ then the value of $\lim_{x\to\infty} \frac{1}{x}\frac{dy}{dx}$ is_

$\boxed{2018-CH}$

Soln.

$$\lim_{x\to\infty} \frac{1}{x}\frac{dy}{dx} = \lim_{x\to\infty} \frac{1}{x} e^{-x^2}(-2x)$$

$$= \lim_{x\to\infty} -2e^{-x^2}$$

$$= 0 \qquad (\because e^{-\infty} = 0)$$

Ans : 0

7) $\lim_{x\to 4} \frac{\sin(x-4)}{x-4} = ?$

$\boxed{2016-CS}$

Soln

Let $x-4 = t$ (or) $t \to 0$

$$\lim_{x\to 4} \frac{\sin(x-4)}{(x-4)} = \lim_{t\to 0} \frac{\sin t}{t} = 1$$

9) $\lim_{x\to\infty} \sqrt{x^2+x-1} - x$ is

2016 - ME

Soln

$$\lim_{x\to\infty} \sqrt{x^2+x-1} - x = \lim_{x\to\infty} x\sqrt{1+1/x-1/x^2} - x$$

$$= \lim_{x\to\infty} \frac{\sqrt{1+1/x-1/x^2} - 1}{1/x} \quad (0/0 \text{ form})$$

$$= \lim_{x\to\infty} \frac{1/2\,(1+1/x-1/x^2)^{-1/2}\,(-1/x^2+2/x^3)}{-1/x^2}$$

$$= \lim_{x\to\infty} \frac{1/2\,(1+1/x-1/x^2)^{-1/2}\,\cancel{(-1/x^2)}\,(1-2/x)}{\cancel{(-1/x^2)}}$$

$$= 1/2\,(1)\,(1) = 1/2$$

Ans: 1/2

(9)

10) $\lim_{x \to 0} x \sin(1/x) =$ ____ [1995]

Soln

Let $x = t \Rightarrow 1/x = 1/t$, $t \to \infty \Rightarrow 1/t \to 0$

$$\lim_{x \to 0} x \sin(1/x) = \lim_{t \to \infty} \frac{\sin t}{t} = 0$$

Ans: 0

11) $\lim_{x \to 0} \frac{e^x - (1 + x + x^2/2)}{x^3} =$ ____ [2007 - ME]

Soln

$$e^x = 1 + x + x^2/2! + x^3/3! + \cdots$$

$$\lim_{x \to 0} \frac{e^x - (1 + x + x^2/2)}{x^3} = \lim_{x \to 0} \frac{x^3/3! + x^4/4! + \cdots + \infty}{x^3}$$

$$= \lim_{x \to 0} \left(1/3! + x/4! + x^2/5! + \cdots \right)$$

$$= 1/3! + 0 = 1/6$$

Ans: 1/6

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